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COMMON FIXED POINT THEOREM OF FOUR MAPPINGS IN CONE METRIC SPACE

XIAN ZHANG

School of Sciences
Jimei University
Xiamen 361021
P. R. China
e-mail: zhangxianjmj@sina.com

Abstract

In this paper, we replace "max" by "co" in contractive condition, obtain a new common fixed point theorem of four mappings in cone metric space. The result unifies and extends many well-known fixed point theorems in metric spaces and cone metric spaces.

1. Introduction

In 1974, Ćirić [2] introduced and studied quasi-contraction mapping in metric space. The well known Ćirić's result is that, see [2, 4, 8]:

Theorem 1. *Let* (X, ρ) *be a complete metric space. A mapping* $T : X \to X$ *such that for some constant* $\lambda \in (0, 1)$ *and for every* $x, y \in X$,

$$\rho(Tx, Ty) \leq \lambda \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(x, Ty), \rho(y, Tx)\}.$$

Then T possesses a unique fixed point.

Guang and Xian [5] replaced the real numbers by an ordered Banach space and defined a cone metric space. They proved some fixed point

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theorems for contractive mappings on cone metric spaces. Ilić and Rakočević [7] generalized Theorem 1 to cone metric space. They proved the following theorem.

Theorem 2. *Let* (X, d) *be a complete cone metric space, with normal cone. A mapping* $T : X \to X$ *such that for some constant* $\lambda \in (0, 1)$ *and for every* $x, y \in X$, *there exists* $s \in C(T, x, y)$ *such that*

$$d(Tx, Ty) \leq \lambda s,$$

where $C(T, x, y) = \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$. *Then T possesses a unique fixed point.*

In [3], Das and Naik extended Theorem 1 to common fixed point of two maps. They proved the following theorem.

Theorem 3. *Let* (X, ρ) *be a complete metric space. Let f be a continuous self-map on X and g be any self-map on X that commutes with f. Further, let f and g satisfy* $g(X) \subseteq f(X)$ *and there exists a constant* $\lambda \in (0, 1)$ *such that for every* $x, y \in X$,

$$\rho(gx, gy) \leq \lambda \max\{\rho(fx, fy), \rho(fx, gx), \rho(fy, gy), \rho(fx, gy), \rho(fy, gx)\}.$$

Then f and g have a unique common fixed point.

In [6], Ilić and Rakočević extended Theorem 3 to cone metric space. They proved the following theorem.

Theorem 4. *Let* (X, d) *be a complete cone metric space with normal cone. Let* $f, g : X \to X$ *be two self-maps on X such that f is continuous and f and g are commuting and* $g(X) \subseteq f(X)$. *Suppose that there exists a constant* $\lambda \in (0, 1)$ *such that for every* $x, y \in X$, *there exists* $s \in C(x, y)$ *such that*

$$d(gx, gy) \leq \lambda s,$$

where $C(x, y) = \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$. *Then f and g have a unique common fixed point.*

In this paper, a new common fixed point theorem is given. The result generalizes corresponding results in metric space and cone metric space.

Now following [5], we give some definitions and auxiliary results.

Let E be a real Banach space and P be a cone in E, we define a partial ordering with respect to P by $x \leq y$, if and only if $y - x \in P$. We shall write $x << y$, if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

Definition 1 [5]**.** Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$, if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space*.

Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. If for every $c \in E$ with $0 << c$, there is N such that for all $n > N$, $d(x_n, x) << c$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$.

Let $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 << c$, there is N such that for all $n, m > N$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence is convergent in X, then X is called a *complete cone metric space*.

A mapping $T : X \to X$ is continuous, if $Tx_n \to Tx$ whenever $x_n \to x \in X$.

Let E be a real linear space, $x \in E$, $A \subseteq E$, and λ be a nonnegative real number. With $x \leq A$ and $x \leq \lambda A$, we denote there is an $s \in A$ such that $x \leq s$ and there is an $s \in A$ such that $x \leq \lambda s$, respectively.

Let E be a real linear space, A be a subset of E, with co$A$, we denote the convex hull of A.

2. Main Theorem

In this section, we shall prove our main theorem.

Theorem 5. *Let* (X, d) *be a complete cone metric space. Let* S, T, I, J *be four mappings from X to X, such that*

(i) $S(X) \subseteq I(X)$ *and* $T(X) \subseteq J(X)$;

(ii) *I and J are continuous*;

(iii) *S, J and T, I are commutative, i.e.,* $SJx = JSx$ *and* $TIx = ITx$ *for all* $x \in X$.

If S, T, I, J *satisfy following contractive condition*:

$$d(Sx, Ty) \leq \lambda co\{0, d(Jx, Iy), d(Jx, Sx), d(Iy, Ty), \frac{1}{2}(d(Jx, Ty) + d(Iy, Sx))\}, \tag{2.1}$$

for all $x, y \in X$. *Then* S, T, I, J *have a unique common fixed point in X.*

Proof. Take an arbitrary $x_0 \in X$, let $y_0 = Jx_0$. Since $S(X) \subseteq I(X)$ and $T(X) \subseteq J(X)$, there are $x_1, x_2 \in X$ such that

$$y_1 := Sx_0 = Ix_1, \; y_2 := Tx_1 = Jx_2.$$

Generally, we have $\{x_n\}$ and $\{y_n\}$, such that for all $n \in \mathbb{N}$

$$y_{2n+1} := Sx_{2n} = Ix_{2n+1}, \; y_{2n+2} := Tx_{2n+1} = Jx_{2n+2}.$$

From the contractive condition (2.1)

$$d(y_1, y_2) = d(Sx_0, Tx_1)$$

$$\leq \lambda co\{0, d(Jx_0, Ix_1), d(Jx_0, Sx_0), d(Ix_1, Tx_1), \frac{1}{2}(d(Jx_0, Tx_1) + d(Ix_1, Sx_0))\}$$

$$= \lambda co\{0, d(y_0, y_1), d(y_0, y_1), d(y_1, y_2), \frac{1}{2}(d(y_0, y_2) + d(y_1, y_1))\}$$

$\Rightarrow$

$$d(y_1, y_2) \leq \lambda co\{0, d(y_0, y_1), d(y_0, y_1), d(y_1, y_2), d(y_0, y_1), d(y_1, y_2)\}$$

$\Rightarrow$

$$d(y_1, y_2) \leq \lambda co\{0, d(y_0, y_1)\}.$$

$$d(y_2, y_3) = d(Sx_2, Tx_1)$$

$$\leq \lambda co\{0, d(Jx_2, Ix_1), d(Jx_2, Sx_2), d(Ix_1, Tx_1), \frac{1}{2}(d(Jx_2, Tx_1) + d(Ix_1, Sx_2))\}$$

$$= \lambda co\{0, d(y_2, y_1), d(y_2, y_3), d(y_1, y_2), \frac{1}{2}(d(y_2, y_2) + d(y_1, y_3))\}$$

$\Rightarrow$

$$d(y_2, y_3) \leq \lambda co\{0, d(y_2, y_1), d(y_2, y_3), d(y_1, y_2), d(y_1, y_2), d(y_2, y_3)\}$$

$$\Rightarrow \qquad d(y_2, y_3) \leq \lambda co\{0, d(y_1, y_2)\},$$

$$\Rightarrow \qquad d(y_2, y_3) \leq \lambda^2 co\{0, d(y_0, y_1)\}.$$

Suppose that

$$d(y_{2n-1}, y_{2n}) \leq \lambda^{2n-1} co\{0, d(y_0, y_1)\},$$

and

$$d(y_{2n}, y_{2n+1}) \leq \lambda^{2n} co\{0, d(y_0, y_1)\}.$$

Then

$$d(y_{2n+1}, y_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leq \lambda co\{0, d(Jx_{2n}, Ix_{2n+1}), d(Jx_{2n}, Sx_{2n}), d(Ix_{2n+1}, Tx_{2n+1}),$$

$$\frac{1}{2}(d(Jx_{2n}, Tx_{2n+1}) + d(Ix_{2n+1}, Sx_{2n}))\}$$

$$= \lambda co\{0, d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}),$$

$$\frac{1}{2}(d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1}))\}$$

$\Rightarrow$

$$d(y_{2n+1}, y_{2n+2}) \leq \lambda co\{0, d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\}$$

$\Rightarrow$

$$d(y_{2n+1}, y_{2n+2}) \leq \lambda co\{0, d(y_{2n}, y_{2n+1})\}$$

$\Rightarrow$

$$d(y_{2n+1}, y_{2n+2}) \leq \lambda^{2n+1} co\{0, d(y_0, y_1)\}.$$

$$d(y_{2n+2}, y_{2n+3}) = d(Sx_{2n+2}, Tx_{2n+1})$$

$$\leq \lambda co\{0, d(Jx_{2n+2}, Ix_{2n+1}), d(Jx_{2n+2}, Sx_{2n+2}), d(Ix_{2n+1}, Tx_{2n+1}), \frac{1}{2}(d(Jx_{2n+2}, Tx_{2n+1}) + d(Ix_{2n+1}, Sx_{2n+2}))\}$$

$$= \lambda co\{0, d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), \frac{1}{2}(d(y_{2n+2}, y_{2n+2}) + d(y_{2n+1}, y_{2n+3}))\}$$

$\Rightarrow$

$$d(y_{2n+2}, y_{2n+3}) \leq \lambda co\{0, d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3})\}$$

$\Rightarrow$

$$d(y_{2n+2}, y_{2n+3}) \leq \lambda co\{0, d(y_{2n+1}, y_{2n+2})\}$$

$\Rightarrow$

$$d(y_{2n+2}, y_{2n+3}) \leq \lambda^{2n+2} co\{0, d(y_0, y_1)\}.$$

Hence, it is proven by induction that for all n

$$d(y_n, y_{n+1}) \leq \lambda^n co\{0, d(y_0, y_1)\}.$$

There are $0 \leq a_n \leq 1$ such that

$$d(y_n, y_{n+1}) \leq \lambda^n a_n d(y_0, y_1).$$

For $0 < m < n$,

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \cdots + d(y_{n-1}, y_n) \\ &\leq (\lambda^m a_m + \cdots + \lambda^{n-1} a_{n-1}) d(y_0, y_1) \\ &\leq \frac{\lambda^m}{1-\lambda} d(y_0, y_1). \end{aligned}$$

For $c \in E$, $0 << c$, there is N such that $\frac{\lambda^m}{1-\lambda} d(y_0, y_1) << c$, for all $m \geq N$. So for $m \geq N$ and all n, we have

$$d(y_m, y_n) << c.$$

Hence, $\{y_n\}$ is a Cauchy sequence. Since X is complete, let $y_n \to y^*$ for some $y^* \in X$.

By the conditions (ii) and (iii)

$$Jy_* = \lim_{n\to\infty} Jy_{2n+1} = \lim_{n\to\infty} JSx_{2n} = \lim_{n\to\infty} SJx_{2n} = \lim_{n\to\infty} Sy_{2n},$$

$$Iy_* = \lim_{n\to\infty} Iy_{2n} = \lim_{n\to\infty} ITx_{2n-1} = \lim_{n\to\infty} TIx_{2n-1} = \lim_{n\to\infty} Ty_{2n-1}.$$

By the contractive condition (2.1), for all n

$$\begin{aligned} &d(Sy_{2n}, Ty_{2n-1}) \\ &\quad \leq \lambda co\{0, d(Jy_{2n}, Iy_{2n-1}), d(Jy_{2n}, Sy_{2n}), d(Iy_{2n-1}, Ty_{2n-1}), \\ &\qquad \frac{1}{2}(d(Jy_{2n}, Ty_{2n-1}) + d(Iy_{2n-1}, Sy_{2n}))\}. \end{aligned}$$

There are real a_n, b_n, c_n, d_n, with $a_n, b_n, c_n, d_n \geq 0$, $a_n + b_n + c_n + d_n \leq 1$, such that

$$d(Sy_{2n}, Ty_{2n-1})$$
$$\leq \lambda(a_n d(Jy_{2n}, Iy_{2n-1}) + b_n d(Jy_{2n}, Sy_{2n}) + c_n d(Iy_{2n-1}, Ty_{2n-1})$$
$$+ d_n \frac{1}{2}(d(Jy_{2n}, Ty_{2n-1}) + d(Iy_{2n-1}, Sy_{2n}))). \tag{2.2}$$

There are subsequences $\{a_{n_m}\}$, $\{b_{n_m}\}$, $\{c_{n_m}\}$, $\{d_{n_m}\}$ of $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ such that $a_{n_m} \to a$, $b_{n_m} \to b$, $c_{n_m} \to c$, $d_{n_m} \to d$ as $m \to \infty$ with $a, b, c, d \geq 0$, $a + b + c + d \leq 1$. Replace the subscript "n" by "n_m" in (2.2) and let $m \to \infty$, we get

$$d(Jy^*, Iy^*) \leq \lambda(a + d)d(Jy^*, Iy^*). \tag{2.3}$$

Since $\lambda(a + d) < 1$, $d(Iy^*, Jy^*) = 0$. Hence, $Iy^* = Jy^*$.

Since

$$d(Sy_*, Ty_{2n-1})$$
$$\leq \lambda co\{0, d(Jy_*, Iy_{2n-1}), d(Jy_*, Sy_*), d(Iy_{2n-1}, Ty_{2n-1}),$$
$$\frac{1}{2}(d(Jy_*, Ty_{2n-1}) + d(Iy_{2n-1}, Sy_*))\},$$

there are real $a'_n, b'_n, c'_n, d'_n \geq 0$, $a'_n + b'_n + c'_n + d'_n \leq 1$, such that

$$d(Sy^*, Ty_{2n-1})$$
$$\leq \lambda(a'_n d(Jy^*, Iy_{2n-1}) + b'_n d(Jy^*, Sy^*) + c'_n d(Iy_{2n-1}, Ty_{2n-1})$$
$$+ d'_n \frac{1}{2}(d(Jy^*, Ty_{2n-1}) + d(Iy_{2n-1}, Sy^*))). \tag{2.4}$$

As the proof of (2.3), there are $a', b', c', d' \geq 0$, $a' + b' + c' + d' \leq 1$, such that

$$d(Sy^*, Iy^*)$$

$$\leq \lambda(a'd(Jy^*, Iy^*) + b'd(Jy^*, Sy^*) + c'd(Iy^*, Iy^*)$$

$$+ d'\frac{1}{2}(d(Jy^*, Iy^*) + d(Iy^*, Sy^*)))$$

$$= \lambda(b' + \frac{1}{2}d')d(Sy^*, Jy^*).$$

Since $\lambda(b' + \frac{1}{2}d') < 1$, $d(Sy^*, Jy^*) = 0$. Hence $Sy^* = Jy^*$. Similarly, $Ty^* = Iy^*$.

From

$$d(Sy_{2n}, y_{2n+2}) = d(Sy_{2n}, Tx_{2n+1})$$

$$\leq \lambda co\{0, d(Jy_{2n}, Ix_{2n+1}), d(Jy_{2n}, Sy_{2n}), d(Ix_{2n+1}, Tx_{2n+1}),$$

$$+ \frac{1}{2}(d(Jy_{2n}, Tx_{2n+1}) + d(Ix_{2n+1}, Sy_{2n}))\}$$

there are real $a_n'', b_n'', c_n'', d_n'' \geq 0$, $a_n'' + b_n'' + c_n'' + d_n'' \leq 1$, such that

$$d(Sy_{2n}, y_{2n+2})$$

$$\leq \lambda(a_n''d(Jy_{2n}, Ix_{2n+1}) + b_n''d(Jy_{2n}, Sy_{2n}) + c_n''d(Ix_{2n+1}, Tx_{2n+1})$$

$$+ d_n''\frac{1}{2}(d(Jy_{2n}, Tx_{2n+1}) + d(Ix_{2n+1}, Sy_{2n}))). \tag{2.5}$$

As the proof of (2.3), there are $a'', b'', c'', d'' \geq 0$, $a'' + b'' + c'' + d'' \leq 1$, such that

$$d(Jy^*, y^*) \leq \lambda(a'' + d'')d(Iy^*, y^*).$$

Since $\lambda(a'' + d'') < 1$, $d(Jy^*, y^*) = 0$. So $Jy^* = y^*$. Therefore, y^* is a common fixed point of S, T, I, J.

Now, if x^* is also a common fixed point of S, T, I, J, then

$$d(x^*, y^*) = d(Sx^*, Ty^*)$$

$$\leq \lambda co\{0,\ d(Jx^*,\ Iy^*),\ d(Jx^*,\ Sx^*),\ d(Iy^*,\ Ty^*),\ \frac{1}{2}(d(Jx^*,\ Ty^*) + d(Iy^*,\ Sx^*))\}$$

$$= \lambda co\{0,\ d(x^*,\ y^*)\}.$$

This implies that $x^* = y^*$. Hence, the common fixed point of S, T, I, J is unique. □

Remark 1. In Theorem 5, let $I = J = id$ (the identity operator on X), or $S = T$, or X be a metric space, we can get many versions of fixed point and common fixed point theorems.

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